THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 1

1 (a) $e^z = e^x \cos y + ie^x \sin y$. Let $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Then we have

$$u_x = e^x \cos y \quad , \quad u_y = -e^x \sin y$$
$$v_x = e^x \sin y \quad , \quad v_y = e^x \cos y$$

Since u_x, u_y, v_x and v_y are continuous and $u_x = v_y$, $u_y = -v_x$ for any $x, y \in \mathbb{R}$, e^z is differentiable for all $z \in \mathbb{C}$.

(b) i. $|e^z| = \sqrt{u^2+v^2} = \sqrt{e^{2x}(\cos^2 y + \sin^2 y)} = e^x$

ii. Since $e^z = e^x e^{iy}$, we have $\arg(z) = y + 2n\pi$, where $n \in \mathbb{Z}$.

iii. Since $|e^z| = e^x > 0$, we have $e^z \neq 0$.

iv. Write $z_j = x_j + iy_j$, where $j = 1, 2$. Then we have

$$\begin{aligned} e^{z_1}e^{z_2} =& e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i \sin y_2) \\ =& e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \sin y_2 \cos y_1)] \\ =& e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) \\ =& e^{z_1+z_2} \end{aligned}$$

v. By 1)a), e^z is differentiable for all $z \in \mathbb{C}$. Furthermore, we have

$$\frac{d}{dz}e^z = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$$

vi. Note that $e^{2\pi i} = e^0(\cos(2\pi) + i\sin(2\pi)) = 1$. By 1)b) iv), $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$.

2 (a) The Euler formula says that $e^{i\theta} = \cos\theta + i\sin\theta$ for any $\theta \in \mathbb{R}$. This implies

$$\frac{e^{ix}+e^{-ix}}{2} = \frac{\cos x + i\sin x + \cos x - i\sin x}{2} = \cos x$$
$$\text{and} \quad \frac{e^{ix}-e^{-ix}}{2i} = \frac{\cos x + i\sin x - \cos x + i\sin x}{2i} = \sin x$$

(b) i. By Chain rule,

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{ie^{iz}+ie^{-iz}}{2i} = \cos z$$
$$\frac{d}{dz}\cos z = \frac{d}{dz}\frac{e^{iz}+e^{-iz}}{2} = \frac{ie^{iz}-ie^{-iz}}{2} = -\sin z$$

ii. $\sin(-z) = \dfrac{e^{-iz}-e^{iz}}{2i} = -\sin z$, $\cos(-z) = \dfrac{e^{-iz}+e^{iz}}{2i} = \cos z$.

iii. R.H.S. $= \cos z + i\sin z = \dfrac{e^{iz}+e^{-iz}}{2} + \dfrac{e^{iz}-e^{-iz}}{2} = e^{iz} =$ L.H.S.

iv. By 1)b)iv), we have

$$e^{i(z_1+z_2)} = e^{iz_1}e^{iz_2}$$

This implies

$$\cos(z_1+z_2) + i\sin(z_1+z_2) = (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\sin z_1 \cos z_2 + \sin z_2 \cos z_1)$$

By comparing the real part and imaginary part on both sides, we get the desired formulas.

v.

$$\begin{aligned}\sin^2 z + \cos^2 z =& (\frac{e^{iz} - e^{-iz}}{2i})^2 + (\frac{e^{iz} + e^{-iz}}{2})^2 \\ =& \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} \\ =& 1\end{aligned}$$

vi.

$$\begin{aligned}\sin z &= \sin(x+iy) \\ &= \frac{e^{ix-y} - e^{-ix+y}}{2i} \\ &= \frac{e^{-y}\cos x + e^{-y} i \sin x - e^{y}\cos x + e^{y} i \sin x}{2i} \\ &= \sin x(\frac{e^{y}+e^{-y}}{2}) + i\cos x(\frac{e^{y}-e^{-y}}{2}) \\ &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

$$\begin{aligned}\cos z &= \cos(x+iy) \\ &= \frac{e^{ix-y} + e^{-ix+y}}{2} \\ &= \frac{e^{-y}\cos x + e^{-y} i \sin x + e^{y}\cos x - e^{y} i \sin x}{2} \\ &= \cos x(\frac{e^{y}-e^{-y}}{2}) - i\sin x(\frac{e^{y}+e^{-y}}{2}) \\ &= \cos x \cosh y - i \sin x \sinh y\end{aligned}$$

vii. By 2)b)vi),

$$\begin{aligned}|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x \cosh^2 y + (1 - \sin^2 x)\sinh^2 y \\ &= \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y \\ &= \sin^2 x + \sinh^2 y\end{aligned}$$

$$\begin{aligned}|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x \cosh^2 y + (1 - \cos^2 x)\sinh^2 y \\ &= \cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y \\ &= \cos^2 x + \sinh^2 y\end{aligned}$$

(c) No. For example, $|\sin(i\log(3))| = |\cos(i\log(3))| = \sqrt{\sinh(\log 3)} = \sqrt{\frac{3-1/3}{2}} > 1.$

(d) By 2)b)vi),

$$\begin{aligned}
\sin z = 0 &\iff |\sin z|^2 = 0 \\
&\iff \sin^2 x + \sinh^2 y = 0 \\
&\iff \sin^2 x = 0 \text{ and } \sinh^2 y = 0 \\
&\iff x = n\pi \text{ and } e^{2y} = 1 \\
&\iff x = n\pi \text{ and } y = 0, \text{ where } n \in \mathbb{Z} \\
&\iff z = n\pi, \text{ where } n \in \mathbb{Z}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\cos z = 0 &\iff |\cos z|^2 = 0 \\
&\iff \cos^2 x = 0 \text{ and } \sinh^2 y = 0 \\
&\iff z = (n + \frac{1}{2})\pi, \text{ where } n \in \mathbb{Z}
\end{aligned}$$

3 (a) By Chain rule,

$$\begin{aligned}
\frac{d}{dz}\cosh z &= \frac{d}{dz}\frac{e^z + e^{-z}}{2} = \frac{e^z - e^{-z}}{2} = \sinh z \\
\frac{d}{dz}\sinh z &= \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z
\end{aligned}$$

(b) $\sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i\frac{e^{iz} - e^{-iz}}{2i} = i\sin z$, $\cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$

$\sin iz = \frac{e^{-z} - e^{z}}{2i} = i\frac{e^{z} - e^{-z}}{2} = i\sinh z \quad , \quad \cos iz = \frac{e^{-z} + e^{z}}{2} = \cosh z$

(c) $\sinh(-z) = \frac{e^{-z} - e^{z}}{2} = -\sinh z$, $\cosh(-z) = \frac{e^{-z} + e^{z}}{2} = \cosh z$

(d) $\cosh^2 z - \sinh^2 z = (\frac{e^z + e^{-z}}{2})^2 - (\frac{e^z - e^{-z}}{2})^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1$

(e) By 2)b)iv) and 3)b), c), we have

$$\begin{aligned}
\sinh(z_1 + z_2) &= -\sinh(i(i(z_1 + z_2))) \\
&= -\,i\sin(i(z_1 + z_2)) \\
&= -\,i(\sin iz_1 \cos iz_2 + \cos iz_1 \sin iz_2) \\
&= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cosh(z_1 + z_2) &= \cosh(i(i(z_1 + z_2))) \\
&= \cos(i(z_1 + z_2)) \\
&= \cos iz_1 \cos iz_2 + \cos iz_1 \cos iz_2 \\
&= \cosh z_1 \cosh z_2 + \cosh z_1 \cosh z_2
\end{aligned}$$

(f) By 2)b)iv) and 3)b), c), we have

$$\begin{aligned}
\sinh z &= -\sinh(i(iz)) \\
&= -i\sin(iz) \\
&= -i\sin(-y+ix) \\
&= -i(\sin(-y)\cosh x + i\cos(-y)\sinh x) \\
&= \sinh x\cos y + i\cosh x\sin y
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cosh z &= \cosh(i(iz)) \\
&= \cos(iz) \\
&= \cos(-y+ix) \\
&= \cosh x\cos(y) + i\sinh x\sin(y)
\end{aligned}$$

(g)

$$\begin{aligned}
|\sinh z|^2 &= \sinh^2 x\cos^2 y + \cosh^2 x\sin^2 y \\
&= \sinh^2 x(1-\sin^2 y) + \cosh^2 x\sin^2 y \\
&= \sinh^2 x + (\cosh^2 x - \sinh^2 x)\sin^2 y \\
&= \sinh^2 x + \sin^2 y
\end{aligned}$$

$$\begin{aligned}
|\cosh z|^2 &= \cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y \\
&= \cosh^2 x\cos^2 y + \sinh^2 x(1-\cos^2 y) \\
&= \sinh^2 x + \cos^2 y(\cosh^2 x - \sinh^2 x) \\
&= \sinh^2 x + \cos^2 y
\end{aligned}$$